FINITE DIMENSIONAL ALGEBRAS AND CELLULAR SYSTEMS

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Abstract. We introduce the notion of a cellular system in order to deal with quasi-hereditary algebras. We shall prove that a necessary and sufficient condition for an algebra to be quasi-hereditary is the existence of a full divisible cellular system. As a further application, we prove that the existence of a full local cellular system is a sufficient condition for a standardly stratified algebra.

To Roger Carter on his 65th birthday

1. Introduction

Quasi-hereditary algebras are an important class of finite dimensional algebras with many applications to Lie theory. Quasi-hereditary algebras have two basic formulations. The ring-theoretic formulation defines a quasi-hereditary algebra through the existence of a heredity chain, while in the module-theoretic formulation it is characterized by the notion of a highest weight category (see [7],[2],[4]). Motivated by the notion of cellular bases and cellular algebras [6], Rui and the author find a third, but incomplete, basic formulation for a quasi-hereditary algebra A over a *splitting* field $\mathtt{k}$ (that is, over a field $\mathtt{k}$ satisfying $\mathrm{End}_A(L) = \mathtt{k}$ for all simple A-module L,) through the existence of a so-called full standard basis, which is a cellular type basis without the involution involved. The purpose of the paper is to complete this formulation for a quasi-hereditary algebra defined over an *arbitrary* field.

A second motivation of the paper is related to a question raised by C.C. Xi. Suppose that A is an algebra over an extension field F of $\mathtt{k}$ and A has a cellular basis over F. If $F \neq \mathtt{k}$, then A may have no cellular basis over $\mathtt{k}$. (In fact, any finite dimensional algebra over a non-splitting field has no cellular basis.) What structure can we say for A over $\mathtt{k}$?

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In this paper, we shall introduce the notion of a cellular system to deal with the issues mentioned above. Roughly speaking, a cellular system is a collection of linear injective transformations from some $\mathtt{k}$-algebras $D(\lambda)$ to A satisfying certain cellular-like axioms. Various conditions imposed on $D(\lambda)$ will give various type of systems such as full cellular systems, divisible cellular systems and local cellular systems. The main result of the paper is to prove that an algebra is quasi-hereditary if and only if A has a full divisible cellular system. We shall also prove that a full local cellular system will give a standard stratification for A, a notion introduced and investigated in the recent work [3] by Cline, Parshall and Scott.

We organize the paper as follows: Cellular systems and their associated standard and costandard modules will be introduced in §2 for algebras defined over a commutative ring $\mathtt{k}$. There are two useful bimodule homomorphisms m_λ and φ_λ which are the main tools of our study. We will discuss them in §3. Full cellular systems are defined to be a system with all φ_λ surjective and will be discussed in §4.

In the last two sections, we shall focus on the algebras defined over a field $\mathtt{k}$. The main result is proved in §5, where we first investigate divisible cellular systems and related representation theory. Finally, we give some further applications to standardly stratified algebras in §6.

Throughout, we assume that $\mathtt{k}$ is a commutative ring with 1. By a $\mathtt{k}$-algebra A (or an algebra over $\mathtt{k}$) we mean that A is an associative algebra with identity element 1, and is finite dimensional if $\mathtt{k}$ is a field. All A-modules will be left modules. A right A-module will be identified with a left A^{op}-module. So an A^{op}-module M^{op} means that M^{op} is a right A-module.

2. Algebras with cellular systems

In this section, we assume that $\mathtt{k}$ is a commutative ring with 1.

Definition 2.1. Let A be a $\mathtt{k}$-algebra and Λ a poset. Suppose that the following conditions hold.

(a) Associated to each $\lambda \in \Lambda$, there are two index sets $I(\lambda), J(\lambda)$ and a $\mathtt{k}$-algebra $D(\lambda)$ with 1, and, associated to each $(i,j) \in I(\lambda) \times J(\lambda)$, there exists a $\mathtt{k}$-linear injective map (not necessarily an algebra homomorphism) $c^\lambda_{i,j} : D(\lambda) \to A$ such that

$$A = \bigoplus_{\lambda\in\Lambda}(\oplus_{i\in I(\lambda),j\in J(\lambda)} c^\lambda_{i,j} D(\lambda)).$$

(b) For any $a \in A$ and $x \in D(\lambda)$, we have

$$\begin{cases} (1) \quad a \cdot c^\lambda_{i,j}(x) \equiv \sum_{i' \in I(\lambda)} c^\lambda_{i',j}(f^\lambda_{i'}(a,i)x) \mod (A^{>\lambda}) \\ (2) \quad c^\lambda_{i,j}(x) \cdot a \equiv \sum_{j' \in J(\lambda)} c^\lambda_{i,j'}(x g^\lambda_{j'}(j,a)) \mod (A^{>\lambda}) \end{cases}$$

where $f^\lambda_{i'}(a,i), g^\lambda_{j'}(j,a) \in D(\lambda)$ are independent of j and i, respectively, and both independent of x, and

$$A^{>\lambda} = \bigoplus_{\mu > \lambda} (\oplus_{i \in I(\mu), j \in J(\mu)} c^\mu_{i,j} D(\mu)). \tag{2.1.1}$$

Then, the system $\mathfrak{c} = \mathfrak{c}(\Lambda; I, J, D) = \{c^\lambda_{i,j}\}_{\lambda,i,j}$ is called a *cellular system* of A defined over the datum $(\Lambda; I, J, D)$.

(c) If, in addition, we assume that every $D(\lambda)$ is free over $\mathtt{k}$, then A is free. Moreover, for a given basis $\delta^\lambda = \{\delta^\lambda_k\}_{k \in K(\lambda)}$ for $D(\lambda)$, we form the set

$$\mathcal{B}^\lambda(\delta^\lambda) = \{c^\lambda_{i,j}(\delta^\lambda_k) \mid i \in I(\lambda), j \in J(\lambda), k \in K(\lambda)\}.$$

Then the union $\mathcal{B}(\delta) = \cup_{\lambda \in \Lambda} \mathcal{B}^\lambda(\delta^\lambda)$ forms a basis for A. Such a basis is called a *generalized cellular type basis* of A.

Examples of cellular systems can be constructed from cellular or cellular type bases as seen from the following result.

Proposition 2.2. *If the* $\mathtt{k}$*-algebra A has a cellular type basis (in the sense of* [5, (1.2.1)]*) and* $\mathtt{k}_1 \subseteq \mathtt{k}$ *is a subring (with the same identity), then A has a cellular system over* $\mathtt{k}_1$.

Proof. Let

$$\{a^\lambda_{i,j} \mid \lambda \in \Lambda, (i,j) \in I(\lambda) \times J(\lambda)\}$$

be a cellular type $\mathtt{k}$-basis for A, where Λ is a poset and $I(\lambda)$ and $J(\lambda)$ are index sets. For each $\lambda \in \Lambda$, let $D(\lambda) = \mathtt{k}$ and define $\mathtt{k}$-linear injective map

$$c^\lambda_{i,j} : D(\lambda) \to A; \quad x \mapsto x a^\lambda_{i,j}, \forall x \in D(\lambda).$$

(Of course, it is $\mathtt{k}_1$-linear.) Then the condition 2.1a is clearly satisfied; while the condition 2.1b follows immediately from the corresponding condition [5, (1.2.2)]. Thus, we have constructed a cellular system for $\mathtt{k}_1$-algebra A. ∎

Conversely, if A has a cellular system, and suppose all $D(\lambda)$ are the same commutative ring R and A is also an R-algebra with all $c^\lambda_{i,j}$ R-linear, then the set $\{c^\lambda_{i,j}(1_R)\}$ is a cellular type basis for A over R.

Definition 2.3. Given a cellular system $\mathfrak{c}(\Lambda; I, J, D)$, there are associated "standard" objects $\Delta(\lambda)$ (resp. $\Delta^{\mathrm{op}}(\lambda)$) in the category A-**mod** of left (resp. right) A-modules with the following properties:

(a) There are injective $\mathtt{k}$-linear maps $a_i^\lambda : D(\lambda) \to \Delta(\lambda)$, $i \in I(\lambda)$ (resp. $b_j^\lambda : D(\lambda) \to \Delta^{\mathrm{op}}(\lambda)$, $j \in J(\lambda)$) such that

$$\Delta(\lambda) = \bigoplus_{i\in I(\lambda)} a_i^\lambda(D(\lambda)) \quad (\text{ resp. } \Delta^{\mathrm{op}}(\lambda) = \bigoplus_{j\in J(\lambda)} b_j^\lambda(D(\lambda))).$$

(b) The module action is given by

$$a \cdot a_i^\lambda(x) = \sum_{i'\in I(\lambda)} a_{i'}^\lambda(f_{i'}^\lambda(a,i)x) \quad (\text{ resp. } b_j^\lambda(x)\cdot a = \sum_{j'\in I(\lambda)} b_{j'}^\lambda(xg_{j'}^\lambda(j,a))),$$

for all $a \in A, i \in I(\lambda), j \in J(\lambda), x \in D(\lambda)$.

Note that, since $A^{>\lambda}\Delta(\lambda) = 0$, $\Delta(\lambda)$ is also an $A/A^{>\lambda}$-module.

(c) If, in addition, we define $D(\lambda)$-module structure on $\Delta(\lambda)$ (resp. $\Delta^{\mathrm{op}}(\lambda)$) naturally by

$$a_i^\lambda(x)\cdot y = a_i^\lambda(xy) \quad (\text{ resp. } y\cdot b_j^\lambda(x) = b_j^\lambda(yx)) \quad \forall x, y \in D(\lambda)$$

Then $\Delta(\lambda)$ (resp. $\Delta^{\mathrm{op}}(\lambda)$) is a A-$D(\lambda)$-bimodule (resp. $D(\lambda)$-A-bimodule) and it is free as a $D(\lambda)$-module.

Let $\nabla(\lambda) = \Delta^{\mathrm{op}}(\lambda)^*$. We shall call the modules $\Delta(\lambda) = \Delta_{\mathfrak{c}}(\lambda)$ and $\nabla(\lambda) = \nabla_{\mathfrak{c}}(\lambda)$ the *standard* and *costandard* objects in the category of A-modules *relative to* the given cellular system $\mathfrak{c}$.

If we define $A^{\geq\lambda}$ similarly as in 2.1.1 with $<$ replaced by $\leq$, we see that the quotient $A^\lambda = A^{\geq\lambda}/A^{>\lambda}$ is an A-A-bimodule and $A^\lambda \cong \oplus_{i,j} c_{i,j}^\lambda(D(\lambda))$ as $\mathtt{k}$-module. Note that A^λ is an ideal of the quotient algebra $A/A^{>\lambda}$.

For fixed $(i,j) \in I(\lambda)\times J(\lambda)$, let $\Delta(\lambda, j)$ (resp. $\Delta(i,\lambda)$) be the $\mathtt{k}$-submodule of A^λ generated by (the image of) $\cup_{i'\in I(\lambda)} c_{i',j}^\lambda(D(\lambda))$ (resp. $\cup_{j'\in J(\lambda)} c_{i,j'}^\lambda(D(\lambda))$). Then, by 2.1b, $\Delta(\lambda, j)$ (resp. $\Delta(i,\lambda)$) is a left (resp. right) A-module, and in the corresponding categories we have A-module isomorphisms

$$\begin{cases} \Delta(\lambda) \cong \Delta(\lambda, j) & \text{via } a_{i'}^\lambda(x) \mapsto c_{i',j}^\lambda(x) + A^{>\lambda}, \\ \Delta^{\mathrm{op}}(\lambda) \cong \Delta(i,\lambda) & \text{via } b_{j'}^\lambda(x) \mapsto c_{i,j'}^\lambda(x) + A^{>\lambda}, \end{cases}$$

for all $i' \in I(\lambda)$, $j' \in J(\lambda)$ and $x \in D(\lambda)$.

The following fact will be useful later on.

Lemma 2.4. *If $c_{i,j}^\mu(x)\Delta(\lambda) \neq 0$, then $\lambda \geq \mu$.*

Proof. The hypothesis implies that some $f^{\lambda}_{i''}(c^{\mu}_{i,j}(x), i')$ is non-zero. This means that $c^{\mu}_{i,j}(x)c^{\lambda}_{i',j'}(1) \not\equiv 0 \mod A^{>\lambda}$. However, $c^{\mu}_{i,j}(x)c^{\lambda}_{i',j'}(1) \in A^{\geq\mu}$. Therefore, $\lambda \geq \mu$. □

3. The homomorphisms m_λ and φ_λ

We shall frequently use two bimodule homomorphisms throughout the paper.

Proposition 3.1. *For any $\lambda \in \Lambda$, there is an A-A-bimodule isomorphism*

$$m_\lambda : \Delta(\lambda) \otimes_{D(\lambda)} \Delta^{\mathrm{op}}(\lambda) \to A^\lambda.$$

Moreover, as left A-module, A^λ is isomorphic to the direct sum $\Delta(\lambda)^{\oplus j_\lambda}$, where $j_\lambda = |J(\lambda)|$.

Proof. Consider $\mathtt{k}$-bilinear map

$$f : \Delta(\lambda) \times \Delta^{\mathrm{op}}(\lambda) \to A^\lambda;\ (a^\lambda_i(x), b^\lambda_j(y)) \mapsto c^\lambda_{i,j}(xy) + A^{>\lambda}$$

for all $(i,j) \in I(\lambda)\times J(\lambda)$ and $x, y \in D(\lambda)$. Clearly, it is $D(\lambda)$-balanced, i.e., $f(az, b) = f(a, zb)$ for all $a \in \Delta(\lambda)$, $b \in \Delta^{\mathrm{op}}(\lambda)$ and $z \in D(\lambda)$. Therefore, f induces a $\mathtt{k}$-linear homomorphism

$$m_\lambda : \Delta(\lambda) \otimes_{D(\lambda)} \Delta^{\mathrm{op}}(\lambda) \to A^\lambda$$

with $m_\lambda(a^\lambda_i(x)\otimes b^\lambda_j(y)) = f(a^\lambda_i(x), b^\lambda_j(y))$. Since the restriction of m_λ to $a^\lambda_i(D(\lambda))\otimes_{D(\lambda)} b^\lambda_j(D(\lambda))$ is a linear isomorphism onto $c^\lambda_{i,j}(D(\lambda)) + A^{>\lambda}$. So m_λ is in fact a linear isomorphism. The fact that m_λ is an A-A-bimodule isomorphism follows from the definition 2.3 of $\Delta(\lambda)$ and $\Delta^{\mathrm{op}}(\lambda)$. The last assertion can be seen from the A-module decomposition

$$\Delta(\lambda) \otimes_{D(\lambda)} \Delta^{\mathrm{op}}(\lambda) = \bigoplus_{j\in J(\lambda)} \Delta(\lambda) \otimes_{D(\lambda)} b^\lambda_j(1).$$

□

Before defining the second homomorphism, we need to look at some "structure constants" with respect to the given cellular system.

Lemma 3.2. *Let $\lambda \in \Lambda$. For any $i, i' \in I(\lambda)$ and $j, j' \in J(\lambda)$, there exists $f^\lambda(j, i') \in D(\lambda)$ such that*

$$c^\lambda_{i,j}(1)c^\lambda_{i',j'}(1) \equiv c^\lambda_{i,j'}(f^\lambda(j,i')) \mod (A^{>\lambda}),$$

where $1 = 1_{D(\lambda)}$. In general, for any $x, y \in D(\lambda)$, we have

$$c^\lambda_{i,j}(x)c^\lambda_{i',j'}(y) \equiv c^\lambda_{i,j'}(xf^\lambda(j,i')y) \mod (A^{>\lambda}).$$

Proof. By 2.1a, we see that $\mathrm{im}(c^\lambda_{i,j}) \cap \mathrm{im}(c^\lambda_{i',j'}) = \{0\}$ for all $(i,j) \neq (i',j')$. So, if

$$c^\lambda_{i,j}(1)c^\lambda_{i',j'}(1) \equiv \sum_{i''\in I(\lambda)} c^\lambda_{i'',j'}(f^\lambda_{i''}(c^\lambda_{i,j}(1),i')) \equiv \sum_{j''\in J(\lambda)} c^\lambda_{i,j''}(g^\lambda_{j''}(j,c^\lambda_{i',j'}(1)))$$

(mod $A^{>\lambda}$), then we have

$$\sum_{i''\in I(\lambda)} c^\lambda_{i'',j'}(f^\lambda_{i''}(c^\lambda_{i,j}(1),i')) - \sum_{j''\in J(\lambda)} c^\lambda_{i,j''}(g^\lambda_{j''}(j,c^\lambda_{i',j'}(1))) = 0,$$

which forces $f^\lambda_{i''}(c^\lambda_{i,j}(1),i') = g^\lambda_{j''}(j,c^\lambda_{i',j'}(1)) = 0$ for all $i'' \neq i$, $j'' \neq j'$ and $f^\lambda_i(c^\lambda_{i,j}(1),i') = g^\lambda_{j'}(j,c^\lambda_{i',j'}(1))$ by the injectivity of $c^\lambda_{i,j}$. Putting

$$f^\lambda(j,i') = f^\lambda_i(c^\lambda_{i,j}(1),i') = g^\lambda_{j'}(j,c^\lambda_{i',j'}(1)), \tag{3.2.1}$$

we proved the first assertion.

Using a similar argument and noting 2.1c, we have

$$c^\lambda_{i,j}(x)c^\lambda_{i',j'}(y) \equiv c^\lambda_{i,j'}(f^\lambda_i(c^\lambda_{i,j}(x),i')y) \equiv c^\lambda_{i,j'}(xg^\lambda_{j'}(j,c^\lambda_{i',j'}(y))) \mod (A^{>\lambda}).$$

Since $c^\lambda_{i,j}(x)c^\lambda_{i',j'}(1) \equiv c^\lambda_{i,j'}(f^\lambda_i(c^\lambda_{i,j}(x),i')) \equiv c^\lambda_{i,j'}(xf^\lambda(j,i'))$, and $c^\lambda_{i,j}$ is injective, we have $f^\lambda_i(c^\lambda_{i,j}(x),i') = xf^\lambda(j,i')$. Similarly, $g^\lambda_{j'}(j,c^\lambda_{i',j'}(y)) = f^\lambda(j,i')y$. Substituting gives the last relation. □

Proposition 3.3. *There is a* $\mathtt{k}$*-linear map*

$$\varphi_\lambda : \Delta^{\mathrm{op}}(\lambda) \otimes_A \Delta(\lambda) \to D(\lambda)$$

such that $\varphi_\lambda(b^\lambda_j(x) \otimes_A a^\lambda_i(y)) = xf^\lambda(j,i)y$. *Moreover,* φ_λ *is a* $D(\lambda)$*-*$D(\lambda)$*-bimodule homomorphism. Hence, the image* $\mathrm{im}(\varphi_\lambda)$ *is an ideal of* $D(\lambda)$ *generated by all the* $f^\lambda(j,i)$.

Proof. Consider the bilinear paring: $\beta_\lambda : \Delta^{\mathrm{op}}(\lambda) \times \Delta(\lambda) \to D(\lambda)$ defined by

$$\beta_\lambda(b^\lambda_j(x), a^\lambda_i(y)) = xf^\lambda(j,i)y.$$

We need to prove that

$$\beta_\lambda(b^\lambda_j(x)a, a^\lambda_i(y)) = \beta_\lambda(b^\lambda_j(x), aa^\lambda_i(y)),$$

for all $a \in A$. By definition, we have

$$\text{l.h.s.} = \beta_\lambda(\sum_{j'\in J(\lambda)} b^\lambda_{j'}(xg^\lambda_{j'}(j,a)), a^\lambda_i(y)) = x(\sum_{j'\in J(\lambda)} g^\lambda_{j'}(j,a)f^\lambda(j',i))y,$$

while

$$\text{r.h.s.} = \beta_\lambda(b^\lambda_j(x), \sum_{i'\in I(\lambda)} a^\lambda_{i'}(f^\lambda_{i'}(a,i))y)) = x(\sum_{i'\in I(\lambda)} f^\lambda(j,i')f^\lambda_{i'}(a,i))y.$$

However, for fixed i_0, j_0, we have

$$\begin{aligned}\sum_{j'\in J(\lambda)} g^\lambda_{j'}(j,a) f^\lambda(j',i) c^\lambda_{i_0,j_0}(1) &\equiv (c^\lambda_{i_0,j}(1)\cdot a) c^\lambda_{i,j_0}(1)\\ &\equiv c^\lambda_{i_0,j}(1)(a\cdot c^\lambda_{i,j_0}(1))\\ &\equiv \sum_{i'\in I(\lambda)} f^\lambda(j,i') f^\lambda_{i'}(a,i) c^\lambda_{i_0,j_0}(1)\\ &\quad \bmod (A^{>\lambda}).\end{aligned}$$

Therefore, $\sum_{j'\in J(\lambda)} g^\lambda_{j'}(j,a) f^\lambda(j',i) = \sum_{i'\in I(\lambda)} f^\lambda(j,i') f^\lambda_{i'}(a,i)$. So β_λ induces a linear map $\varphi_\lambda : \Delta^{\mathrm{op}}(\lambda)\otimes_A \Delta(\lambda) \to D(\lambda)$. It is easy to see that φ_λ is a $D(\lambda)$-$D(\lambda)$-bimodule homomorphism, proving the last assertion. □

There is a nice relation between m_λ and φ_λ. Recall from definition 2.3 that $\Delta(\lambda)$ is an $A/A^{>\lambda}$-module.

Proposition 3.4. *If $a, a' \in \Delta(\lambda)$ and $b \in \Delta^{\mathrm{op}}(\lambda)$ then*

$$m_\lambda(a\otimes_{D(\lambda)} b)a' = a\varphi_\lambda(b\otimes_A a'),$$

where m_λ is the A-A-bimodule isomorphism given in 3.1.

Proof. It suffices to check the equality for $a = a^\lambda_i(1), a' = a^\lambda_{i'}(1)$ and $b = b^\lambda_j(1)$. We leave this to the reader. □

4. Full cellular systems

The homomorphism φ_λ reflects nice structure of the algebra as we will see in the following results.

Proposition 4.1. *Let A be a $\mathtt{k}$-algebra with a cellular system $\mathfrak{c} = \mathfrak{c}(\Lambda; I, J, D) = \{c^\lambda_{i,j}\}_{\lambda,i,j}$, and let $\lambda\in\Lambda$. Then the following two statements are equivalent:*
(a) the ideal A^λ of $A/A^{>\lambda}$ is idempotent;
(b) $\mathrm{im}(\varphi_\lambda) = D(\lambda)$.

Proof. Let $J = A^\lambda$. Then J^2 is generated as $\mathtt{k}$-module by all $c^\lambda_{i,j}(x)c^\lambda_{i',j'}(y)$ for all i, j, i', j' and $x, y \in D(\lambda)$. Since $c^\lambda_{i,j}(x)c^\lambda_{i',j'}(y) = c^\lambda_{i,j'}(xf^\lambda(j,i')y)$ (see 3.2), if $J^2 = J$ then all $c^\lambda_{i,j'}(xf^\lambda(j,i')y)$, where $j\in J(\lambda)$, $i'\in I(\lambda)$ and $x, y\in D(\lambda)$, must span $c^\lambda_{i,j'}(D(\lambda))$. This means that the ideal generated by all $f^\lambda(j,i')$, which is $\mathrm{im}(\varphi_\lambda)$, must equal $D(\lambda)$, since $c^\lambda_{i,j}$ is injective. Thus, we have seen that $J^2 = J$ if and only if $\mathrm{im}(\varphi_\lambda) = D(\lambda)$. □

We say that a cellular system is *full* if the condition (b) in 4.1 holds for every $\lambda \in \Lambda$. So a full system gives idempotent ideals which in many cases have nice homological properties (see [1]). Note that two classes of idempotent ideals — heredity and stratifying ideals — are used in the study of quasi-hereditary and stratifying algebras.

If φ_λ send a 'copy' of $\Delta^{\mathrm{op}}(\lambda)$ onto $D(\lambda)$, (Note that this condition is stronger than the onto condition $\mathrm{im}(\varphi_\lambda) = D(\lambda)$.) then $\Delta(\lambda)$ will be a cyclic module.

Lemma 4.2. *For any $v \in \Delta(\lambda)$, if $\varphi_\lambda(\Delta^{\mathrm{op}}(\lambda) \otimes_A v) = D(\lambda)$, then $\Delta(\lambda) = Av = A^\lambda v$.*

Proof. By 3.3 and 3.4, we have

$$\begin{aligned}\Delta(\lambda) \supseteq Av \supseteq A^\lambda v &= m_\lambda(\Delta(\lambda) \otimes_{\mathtt{k}} \Delta^{\mathrm{op}}(\lambda))v \\ &= \Delta(\lambda)\varphi_\lambda(\Delta^{\mathrm{op}}(\lambda) \otimes_A v) = \Delta(\lambda),\end{aligned}$$

since $\varphi_\lambda(\Delta^{\mathrm{op}}(\lambda) \otimes_A v) = D(\lambda)$. Therefore, $\Delta(\lambda) = Av = A^\lambda v$. □

We can know more properties of the modules $\Delta(\lambda)$ in this case.

Proposition 4.3. *If $\varphi_\lambda(\Delta^{\mathrm{op}}(\lambda) \otimes_A v) = D(\lambda)$ for some $v \in \Delta(\lambda)$ then*
(a) $Hom_A(\Delta(\lambda), \Delta(\mu)) = 0$ unless $\lambda \leq \mu$,
(b) $Hom_A(\Delta(\lambda), \Delta(\lambda)) \cong D(\lambda)$ as $\mathtt{k}$-algebra.

Proof. By the hypothesis and 4.2, we have $\Delta(\lambda) = Av = A^\lambda z$. Thus, $\mathrm{Hom}_A(\Delta(\lambda), \Delta(\mu)) \neq 0$ implies that there exist $a \in A^\lambda$ and $f \in \mathrm{Hom}_A(\Delta(\lambda), \Delta(\mu))$ such that $af(v) = f(av) \neq 0$, which implies $\lambda \leq \mu$, by 2.4.

Let $\lambda = \mu$. For any $f \in \mathrm{End}_A(\Delta(\lambda))$, f is determined by $v' = f(v)$ since $\Delta(\lambda) = A^\lambda v$. On the other hand, the hypothesis implies that there exists an element $w \in \Delta^{\mathrm{op}}(\lambda)$ such that $\varphi_\lambda(w \otimes_A v) = 1$. So, by 3.4, we have

$$\begin{aligned}f(v) &= f(v\varphi_\lambda(w \otimes_A v)) = f(m_\lambda(v \otimes w)v) \\ &= m_\lambda(v \otimes w)v' = v\varphi_\lambda(w \otimes_A v').\end{aligned}$$

That is, every such a homomorphism is a right multiplication by an element of $D(\lambda)$. Since, for $x \in D(\lambda)$, right multiplication by x on the elements of $\Delta(\lambda)$ defines a left A-module homomorphism on $\Delta(\lambda)$, we see that $D(\lambda) = \{\varphi_\lambda(w \otimes_A f(v)) \mid f \in \mathrm{End}_A(\Delta(\lambda))\}$, and the map $\theta : f \mapsto \varphi_\lambda(w \otimes_A f(v))$ gives a $\mathtt{k}$-linear isomorphism between $\mathrm{End}_A(\Delta(\lambda))$ and $D(\lambda)$. Now, for $f, g \in \mathrm{End}_A(\Delta(\lambda))$, write $f(v) = av$,

for some $a \in A$. Then,

$$\begin{aligned}\theta(g \circ f) &= \varphi_\lambda(w \otimes_A g(f(v))) = \varphi_\lambda(w \otimes_A ag(v)) \\ &= \varphi_\lambda(w \otimes_A av\varphi_\lambda(w \otimes_A g(v))) \\ &= \varphi_\lambda(w \otimes_A f(v))\varphi_\lambda(w \otimes_A g(v)).\end{aligned}$$

Therefore, θ is an algebra isomorphism from $\mathrm{End}_A(\Delta(\lambda))$ to $D(\lambda)^{\mathrm{op}}$. □

5. Divisible cellular systems

A cellular system $\mathfrak{c}(\Lambda; I, J, D)$ of A is said to be *divisible*, if every $D(\lambda)$ is a division ring. In this case, the ring $\mathtt{k}$ must be a field. So, *in this section, we assume that* $\mathtt{k}$ *is a field.*

Let A be a finite dimensional algebra over $\mathtt{k}$, and let A-**mod** be the category of finite dimensional A-modules. Recall that, for a given A-module M, the head $\mathrm{hd}(M)$ of M is the largest semi-simple quotient module of M and the radical of M is the submodule $\mathrm{rad}(M)$ of M such that $M/\mathrm{rad}(M) = \mathrm{hd}(M)$. If L is a simple A-module, let $[M : L]$ be the multiplicity of L in M as a composition factor.

We point out that the ideas behind the proofs of Theorems 5.1, 5.2 and 5.3 are similar to the counter-part given in [5, (2.4),(4.2)]. For completeness, we give the details of the proofs so that one may see some difference with the bimodule structure defined in 2.3c under consideration.

Theorem 5.1. *Let A be a finite dimensional $\mathtt{k}$-algebra with a divisible cellular system $\mathfrak{c}(\Lambda; I, J, D)$ and let $\Lambda_{\mathrm{irr}} = \{\lambda \in \Lambda \mid \varphi_\lambda \neq 0\}$.*

(a) For any $\lambda \in \Lambda_{\mathrm{irr}}$, we have

$$\mathrm{rad}(\Delta(\lambda)) = \{x \in \Delta(\lambda) \mid \varphi_\lambda(\Delta^{\mathrm{op}}(\lambda) \otimes_A x) = 0\}.$$

and $L(\lambda) := \Delta(\lambda)/\mathrm{rad}(\Delta(\lambda))$ is simple.

(b) Let $\lambda \in \Lambda_{\mathrm{irr}}$. If $L(\lambda)$ is a composition factor of $\Delta(\mu)$ then $\lambda \leq \mu$ and $[\Delta(\lambda) : L(\lambda)] = 1$.

(c) $\{L(\lambda) \mid \lambda \in \Lambda_{\mathrm{irr}}\}$ is a complete set of all non-isomorphic simple A-modules.

Proof. Let $N_\lambda = \{x \in \Delta(\lambda) \mid \varphi_\lambda(\Delta^{\mathrm{op}}(\lambda) \otimes_A x) = 0\}$. Then N_λ is an A-submodule of $\Delta(\lambda)$ and $\lambda \in \Lambda_{\mathrm{irr}}$ if and only if $N_\lambda \neq \Delta(\lambda)$. If $0 \neq \bar{v} = v + N_\lambda \in \Delta(\lambda)/N_\lambda$, then $\varphi_\lambda(\Delta^{\mathrm{op}}(\lambda) \otimes_A v) \neq 0$, and hence $\varphi_\lambda(\Delta^{\mathrm{op}}(\lambda) \otimes_A v) = D(\lambda)$ since $D(\lambda)$ is a division ring and $\varphi_\lambda(\Delta^{\mathrm{op}}(\lambda) \otimes_A v)$ is a right ideal of $D(\lambda)$. Thus, $\varphi_\lambda(w \otimes_A v) = 1$ for some w and so $x = x\varphi_\lambda(w \otimes_A v) = m_\lambda(x \otimes w)v \in Av$ for all $x \in \Delta(\lambda)$. This shows that $\Delta(\lambda) = Av$ and $\Delta(\lambda)/N_\lambda = A\bar{v}$. So $\Delta(\lambda)/N_\lambda$ is generated by its

any non-zero element. Hence it is a simple A-module and consequently, $N_\lambda \supseteq \operatorname{rad}(\Delta(\lambda))$.

If $\operatorname{rad}(\Delta(\lambda)) \neq N_\lambda$ then there is a simple module L in $\operatorname{hd}(\Delta(\lambda))$ such that the projection η from $\Delta(\lambda)$ onto L does not map N_λ to zero. Therefore $\eta(N_\lambda) = L = \eta(\Delta(\lambda))$. Thus, we have $\eta(v) = \eta(u)$ for some $u \in N_\lambda$. Since $\varphi_\lambda(w \otimes_A u) = 0$, we have

$$\begin{aligned}\eta(v) = \eta(v\beta_\lambda(w \otimes_A v)) &= m_\lambda(v \otimes w)\eta(v)\\ &= \eta(m(v \otimes v)u) = \eta(v\varphi_\lambda(w \otimes_A u)) = 0.\end{aligned}$$

So $\eta(\Delta(\lambda)) = 0$, a contradiction. Therefore, $\operatorname{rad}\Delta(\lambda) = N_\lambda$, proving (a).

We now prove (b). If $L(\lambda)$ is a composition factor of $\Delta(\mu)$ then there is an A-homomorphism $\eta : \Delta(\lambda) \to \Delta(\mu)/N$ for some A-submodule $N \subset \Delta(\mu)$ such that $\operatorname{im}(\eta) \cong L(\lambda)$. Since $\lambda \in \Lambda_{\text{irr}}$, we have $\Delta(\lambda) = A^\lambda v$ for some $v \in \Delta(\lambda)$. Thus, $\eta \neq 0$, i.e., $a\eta(v) \neq 0$ for some $a \in A^\lambda$, implies that there exist $c^\lambda_{i,j}(1)$ and $a^\mu_k(1)$ such that $c^\lambda_{i,j}(1)a^\mu_k(1) \notin N$, forcing $\lambda \leq \mu$ by 2.4.

If $\mu = \lambda$, we claim that η is surjective. Indeed, suppose $\eta(v) = v' + N$. Then

$$\begin{aligned}\eta(v) &= \eta(v\varphi_\lambda(w \otimes_A v)) = \eta(m_\lambda(v \otimes w)v)\\ &= m_\lambda(v \otimes w)v' + N = v\varphi_\lambda(w \otimes_A v') + N.\end{aligned}$$

Since $\varphi_\lambda(w \otimes_A v') \neq 0$, η is surjective. By the claim, we have $\Delta(\lambda)/N \cong L(\lambda)$, and hence, $[\Delta(\lambda) : L(\lambda)] = 1$.

To prove (c), we first note from the argument above that $L(\lambda) \cong L(\mu)$ implies $\lambda = \mu$. We now prove that, for any simple A-module L, $L \cong L(\lambda)$ for some $\lambda \in \Lambda_{\text{irr}}$. We order Λ linearly: $\mu = \lambda_1, \lambda_2, \ldots, \lambda_n$ such that $\lambda_i \geq \lambda_j$ implies $i \leq j$, and define $J_k = \bigoplus_{r=1}^k (\oplus_{i,j} c^{\lambda_r}_{i,j}(D(\lambda_r))$. Then, we have a chain of ideals of A

$$0 = J_0 \subset J_1 \subset \cdots \subset J_m = A \tag{5.1.1}$$

Let $f : A \to L$ be an epimorphism, and let J_i be the minimal ideal in the above filtration such that $f|_{J_i} \neq 0$. Then f induces an epimorphism $f : A^{\lambda_i} \cong J_i/J_{i-1} \to L$. Since $J_{i-1}L = 0$ and $J_iL \neq 0$, it follows that $(A^{\lambda_i})^2 \neq 0$, which implies $\varphi_{\lambda_i} \neq 0$. Therefore, $\lambda_i \in \Lambda_{\text{irr}}$. Now, since $A^{\lambda_i} \cong \Delta(\lambda_i)^{\oplus j_{\lambda_i}}$ (3.1), we obtain an epimorphism $f : \Delta(\lambda_i) \to L$. Therefore, by (b), $L(\lambda_i) \cong L$. □

Let $\Delta = \{M \in \operatorname{Ob}(A\text{-}\mathbf{mod}) \mid M \cong \Delta(\lambda) \text{ for some } \lambda \in \Lambda\}$. A Δ-filtration of a module M is a sequence of submodules:

$$0 = M_0 \subseteq M_1 \subseteq \cdots \subseteq M_n = M$$

such that $M_i/M_{i-1} \in \Delta$.

Theorem 5.2. *Let A be a finite dimensional* $\mathtt{k}$*-algebra with a divisible cellular system* $\mathfrak{c}(\Lambda; I, J, D)$. *Then the projective cover* $P(\lambda)$ *of* $L(\lambda)$ *with* $\lambda \in \Lambda_{\rm irr}$ *has a* Δ*-filtration. If* $[P(\lambda) : \Delta(\mu)]$ *denotes the number of sections isomorphic to* $\Delta(\mu)$ *in such a filtration, then* $[P(\lambda) : \Delta(\mu)] \neq 0 \Rightarrow \mu \geq \lambda$ *and* $[P(\lambda) : \Delta(\lambda)] = 1$.

Proof. Let $P = P(\lambda)$, and let i be the minimal index in the sequence 5.1.1 such that $J_i P = P$. Then $J_i = M \oplus A^{\geq \lambda_i}$ where

$$M = \bigoplus_{j \leq i, \lambda_j \not\geq \lambda_i} (\oplus_{r,s} c_{r,s}^{\lambda_j}(D(\lambda_j)).$$

is an ideal contained in J_i, and $P = J_i P = MP \oplus A^{\geq \lambda_i} P = A^{\geq \lambda_i} P$ since P is indecomposable and $MP = 0$ by 2.4. For any $\nu \in \Lambda$, put $A^\nu P = A^{\geq \nu} P / A^{> \nu} P$. Then, the above argument shows that λ_i is the minimal ν (w.r.t. the partial ordering on Λ) with $A^\nu P \neq 0$. In particular, $\lambda_i \leq \lambda$ since we certainly have $A^\lambda P \neq 0$. On the other hand, since $A^{\lambda_i} P \cong A^{\lambda_i} \otimes_A P$ is a homomorphic image of P. We have

$$\begin{aligned}
\mathrm{Hom}_A(P, \Delta(\lambda_i)) &\supseteq \mathrm{Hom}_A(A^{\lambda_i} \otimes_A P, \Delta(\lambda_i)) \\
&\cong \mathrm{Hom}_{D(\lambda_i)}(\Delta^{\rm op}(\lambda_i) \otimes_A P, \mathrm{End}_A(\Delta(\lambda_i)) \quad \text{by 3.1} \\
&\cong \mathrm{Hom}_{D(\lambda_i)}(\Delta^{\rm op}(\lambda_i) \otimes_A P, D(\lambda_i)) \quad \text{by 3.3} \\
&\neq 0, \quad \text{since } \Delta^{\rm op}(\lambda_i) \otimes_A P \neq 0.
\end{aligned}$$

So $L(\lambda)$ is a composition factor of $\Delta(\lambda_i)$. By 5.1 we have $\lambda \leq \lambda_i$. Therefore, $\lambda_i = \lambda$ and $P = A^{\geq \lambda} P$. Now, $A^{\geq \lambda}$ has a filtration with sections A^μ, $\mu \geq \lambda$. It follows that P has a filtration with sections isomorphic to $A^\mu \otimes_A P \cong \Delta(\mu) \otimes_{D(\mu)} (\Delta^{\rm op}(\mu) \otimes_A P)$. Therefore, P has a Δ-filtration whose sections $\Delta(\mu)$ satisfies $\mu \geq \lambda$. Finally, since $\dim_{D(\lambda)}(\Delta^{\rm op}(\lambda) \otimes_A P) = \dim_{D(\lambda)} \mathrm{Hom}_{D(\lambda)}(\Delta^{\rm op}(\lambda) \otimes_A P, D(\lambda)) \leq \dim_{D(\lambda)} \mathrm{Hom}_A(P(\lambda), \Delta(\lambda)) = 1$, we have $A^\lambda \otimes_A P(\lambda) \cong \Delta(\lambda)$, proving $[P(\lambda) : \Delta(\lambda)] = 1$. □

We are now ready to prove the main result of the paper.

Theorem 5.3. *Let A be a finite dimensional algebra over a field* $\mathtt{k}$. *Then A is quasi-hereditary if and only if A has a divisible cellular system* $\mathfrak{c}(\Lambda; I, J, D)$ *with* $\Lambda = \Lambda_{\rm irr}$. *The latter is equivalent to that A has a full divisible cellular system*

Proof. The "if" part follows from [2, Theorem 3.6] since, by Theorems 5.1 and 5.2, the category A-**mod** is a highest weight category. It is also easy to check directly that the sequence 5.1.1 is a heredity chain in this case.

Conversely, suppose now that A is quasi-hereditary. Then, there is a poset Λ on which the highest weight category A-**mod** is defined. We now construct a divisible cellular system by induction on Λ.

If $\Lambda = \{\lambda\}$ has a single element, then A itself is a heredity ideal. So there exists a primitive idempotent e of A such that $A \cong Ae \otimes_{eAe} eA$. Since Ae is projective indecomposable, eAe is a division ring. Thus, Ae (resp. eA) is a right (resp. left) vector space over eAe. Choose eAe-bases $\{a_i\}_{i\in I}$ and $\{b_j\}_{j\in J}$ for Ae and eA, respectively, and define

$$c_{i,j} : eAe \to A$$

sending $x \in eAe$ to $c_{i,j}(x) = a_i^\lambda x \otimes_{eAe} b_j^\lambda$. Putting $D = eAe$, one obtains a cellular system $\mathfrak{c}(\Lambda; I, J, D) = \{c_{i,j}\}_{i,j}$. Clearly, it is divisible and $\Lambda_{\text{irr}} = \Lambda$.

Assume $|\Lambda| > 1$ and let $\mu \in \Lambda$ be a maximal element. We order Λ linearly: $\mu = \lambda_1, \lambda_2, \ldots, \lambda_n$ such that $\lambda_i \geq \lambda_j$ implies $i \leq j$ and $\{\lambda_2, \cdots, \lambda_t\} = \{\nu \in \Lambda : \nu \not\leq \mu\}$. Then, there is a heredity chain of A

$$0 = J_0 \subset J = J_1 \subset \cdots \subset J_m = A,$$

such that $J_i/J_{i-1} \cong A_i f \otimes_{fA_if} fA_i$ for some (primitive) idempotent f of $A_i := A/A_{i-1}$, and ${}_A(A_i f) \cong \Delta(\lambda_i)$, the standard object corresponding to λ_i.

Let $B = A/J$ and $\pi : A \to B$ the natural epimorphism. Then B is a quasi-hereditary algebra with the weight poset $\Lambda_1 = \Lambda\backslash\{\mu\}$. By induction, B has a divisible cellular system $\mathfrak{d} = \mathfrak{d}(\Lambda_1, I, J, D) = \{d_{i,j}^\lambda\}_{\lambda,i,j}$ with the property

$$J_r/J = (\oplus_{i,j} d_{i,j}^{\lambda_2} D(\lambda_2)) \oplus \cdots \oplus (\oplus_{i,j} d_{i,j}^{\lambda_r} D(\lambda_r)), \ \ \forall r \in [2, m].$$

We now inductively construct a cellular system $\{c_{i,j}^{\lambda_k}\}$ for A. Since $J = J_1$ is a heredity ideal of A, there is an idempotent e of A such that $J \cong Ae \otimes_{eAe} eA$ and $Ae \cong \Delta(\mu)$ is a projective indecomposable module. So eAe is a division algebra. We extend the map D from Λ_1 to Λ by setting $D(\mu) = eAe$. As in the discussion for $|\Lambda| = 1$, we may define injective linear maps $c_{i,j}^\mu : D(\mu) \to J$, where $i \in I_\mu$ and $j \in J_\mu$ for some index sets I_μ, J_μ, such that $J = \oplus_{i,j} c_{i,j}^\mu(D(\mu))$. Now, the maps I and J are also extended by setting $I(\mu) = I_\mu$ and $J(\mu) = J_\mu$.

Since $\lambda_k \not\leq \mu$, $2 \leq k \leq t$, we have $J_t \cong J_t/J \oplus J$. So J_t/J can be viewed as an ideal of A, and we may define $c_{i,j}^{\lambda_k} = d_{i,j}^{\lambda_k}$ for $2 \leq k \leq t$.

Assume now $k > t$. Using the linear injections $d_{i,j}^{\lambda_k} : D(\lambda_k) \to J_k/J$, we may easily define linear injective maps $c_{i,j}^{\lambda_k} : D(\lambda_k) \to J_k$ such that

$\pi \circ c_{i,j}^{\lambda_k} = d_{i,j}^{\lambda_k}$ and the following diagram commute:

$$\begin{array}{ccc} D(\lambda_k) & \xrightarrow{c_{i,j}^{\lambda_k}} & J_k \\ \downarrow{\scriptstyle d_{i,j}^{\lambda_k}} & & \downarrow{\scriptstyle \pi_k} \\ J_k/J & \xrightarrow{\bar{\pi}_k} & J_k/J_{k-1} \end{array}$$

where $\bar{\pi}_k$ and π are natural maps. Note that, since $\bar{\pi}_k(d_{i,j}^{\lambda_k}(D(\lambda_k)) = J_k/J_{k-1}$, we have $\pi_k(c_{i,j}^{\lambda_k}(D(\lambda_k)) = J_k/J_{k-1}$. Therefore, one sees easily that $A = \oplus_{\lambda,i,j} c_{i,j}^{\lambda}(D(\lambda))$ as vector space.

It remains to prove that the system $\{c_{i,j}^{\lambda}\}_{\lambda,i,j}$ satisfies the conditions in 2.1b. The conditions hold clearly for $\lambda \in \{\lambda_1, \cdots, \lambda_t\}$. Since $\{d_{i,j}^{\lambda}\}$ is a cellular system of B, we have for $\lambda = \lambda_k$ with $k > t$, $a \in A$ and $x \in D(\lambda)$,

$$\begin{aligned} \pi(a) \cdot d_{i,j}^{\lambda}(x) &\equiv \sum_{i' \in I(\lambda)} d_{i',j}^{\lambda}(f_{i'}^{\lambda}(\pi(a), i)x) \mod (B^{>\lambda}) \\ &\equiv \sum_{i' \in I(\lambda)} d_{i',j}^{\lambda}(f_{i'}^{\lambda}(\pi(a), i)x) \mod (J_{k-1}/J). \end{aligned}$$

Thus, from the definition of $c_{i,j}^{\lambda}$ and noting $J \subset A^{>\lambda}$, we have

$$\begin{aligned} a \cdot c_{i,j}^{\lambda}(x) &\equiv \sum_{i' \in I(\lambda)} c_{i',j}^{\lambda}(f_{i'}^{\lambda}(\pi(a), i)x) \mod (J_{k-1}) \\ &\equiv \sum_{i' \in I(\lambda)} c_{i',j}^{\lambda}(f_{i'}^{\lambda}(\pi(a), i)x) \mod (A^{>\lambda}), \end{aligned}$$

where the $f_{i'}^{\lambda}(\pi(a), i)$ is independent of j and x. By a symmetric argument, we see that a similar relation holds for $c_{i,j}^{\lambda}(x) \cdot a$. Therefore, we have obtained a cellular system which is clearly divisible and $\Lambda = \Lambda_{\text{irr}}$. $\square$

6. Local cellular systems and standard stratifications

We shall assume again in this section that $\mathtt{k}$ is a field and A is finite dimensional over A.

A cellular system $\mathfrak{c}(\Lambda; I, J, D)$ of A is said to be *local*, if every $D(\lambda)$ is a local ring.

Recall from [3, (2.1)] that an ideal of A is called a (left) *standard stratifying ideal* if $J = AeA$ for some idempotent $e \in A$ and ${}_AJ$ is a projective A-module. The algebra A is said to be (left) *standardly stratified* of length n if A has a chain

$$0 = J_0 \subset J_1 \subset \cdots \subset J_n = A$$

of ideals such that J_i/J_{i-1} is a standard stratifying ideal in A/J_{i-1} ($i = 1, 2, \cdots, n$). In this case, the chain is called a (left) *standard stratification* of A. If the condition that ${}_AJ$ is projective is replaced by that J_A is projective, then we obtain the notion of a right standard stratifying ideal, etc..

Theorem 6.1. *Let A be a finite dimensional* $\mathtt{k}$*-algebra. Suppose that A has a local cellular system $\mathfrak{c}(\Lambda; I, I, D)$ which is full, i.e.,* $\operatorname{im}(\varphi_\lambda) = D(\lambda)$ *for all $\lambda \in \Lambda$. Then A is (left and right) standardly stratified of length $|\Lambda|$.*

Proof. We order Λ linearly: $\lambda_1, \lambda_2, \ldots, \lambda_n$ such that $\lambda_i \geq \lambda_j$ implies $i \leq j$ and define

$$J_k = \bigoplus_{r=1}^{k} (\oplus_{i\in I(\lambda), j\in J(\lambda)} c_{i,j}^{\lambda_r} D(\lambda_r)).$$

Then, $A^{\geq \lambda_k} \subseteq J_k$, and we obtain a chain of ideals

$$0 = J_0 \subset J_1 \subset \cdots \subset J_m = A. \tag{6.1.1}$$

We now prove that this chain is a standard stratification.

Since each $D(\lambda)$ is local and $\operatorname{im}(\varphi_\lambda) = D(\lambda)$, there exists at least one $z = f^\lambda(j, i)$, which is not in the unique maximal ideal of $D(\lambda)$. Hence, z is invertible. Thus,

$$c_{i,j}^\lambda(z^{-1})^2 \equiv c_{i,j}^\lambda(z^{-1} f^\lambda(j, i) z^{-1}) \equiv c_{i,j}^\lambda(z^{-1}) \mod (A^{>\lambda}).$$

Let e be the image of $c_{i,j}^\lambda(z^{-1})$ in $\bar{A} = A/J_{k-1}$, where $\lambda = \lambda_k$. Then e is an idempotent of $\bar{A}$ and $J_k/J_{k-1} = \bar{A}e\bar{A}$ has a basis consisting of the images of the elements of the form $c_{i',j'}^\lambda(x) = c_{i',j}^\lambda(x) e c_{i,j'}^\lambda(z^{-1})$. Therefore, $\bar{A}e \cong \Delta(\lambda)$ and J_k/J_{k-1} is a (left and right) projective $\bar{A}$-module by 3.1. Consequently, it is a standard stratifying ideal and the chain 6.1.1 is a standard stratification of A. $\square$

We remark that, by the theorem, each $\Delta(\lambda)$ is a projective indecomposable $A/A^{>\lambda}$-module, but its simple head may have multiplicity more than 1.

From the above proof, we have immediately the following result.

Theorem 6.2. *Let A be a finite dimensional* $\mathtt{k}$*-algebra. Suppose that A has a cellular system $\mathfrak{c}(\Lambda; I, I, D)$ with the following property: For any $\lambda \in \Lambda$, there exists $(i, j) \in I(\lambda) \times J(\lambda)$ such that $f^\lambda(j, i)$ is invertible in $D(\lambda)$. Then A is (left and right) standardly stratified.*

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